WHAT IS ALGEBRAIC GEOMETRY? FOR THE LAYMAN

This is a document made with the sole purpose of explaining the titular question at the level of someone who does not know much mathematics beyond calculus.

What is algebraic geometry about?

In order to answer the question in the title of this section, it may be useful to think about how one would answer it if "algebraic geometry" were replaced by another field of mathematics. For example, how would one answer the question "what is linear algebra about?". One could say "linear algebra is the study of vector spaces", but this seems to miss the mark: when one learns linear algebra, one realizes that we are not only interested in vector spaces, but also maps between them, so-called "linear maps". What this example shows is that in order to answer the question "what is algebraic geometry about?", we should not only specify the objects of study in algebraic geometry, but also maps between such objects.

In a word, algebraic geometry is the study of algebraic varieties and morphisms between them. This answer is useless if one does not know the definitions of "algebraic variety" and "morphism of algebraic varieties", so defining these terms is what I will do now.

Varieties

To keep things simple, suppose we are working over the real numbers $\mathbf{R}$. We can form the ring $\mathbf{R}[x]$, the ring of polynomials in one variable with coefficients in $\mathbf{R}$. Elements in $\mathbf{R}[x]$ are of the form

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \quad a_i \in \mathbf{R},\ 0 \leq i \leq n,$$

for some natural number n. More generally, we can consider the ring of polynomials in n variables with coefficients in $\mathbf{R}$, denoted $\mathbf{R}[x_1, x_2, \ldots, x_n]$. Let $\mathbf{R}^n$ denote the set of n-tuples of elements in $\mathbf{R}$ and let $f(x_1, \ldots, x_n) \in \mathbf{R}[x_1, \ldots, x_n]$ (the symbol "$\in$" should be read as "is an element of"). We can look at the following set:

$$Z(f) := \{(p_1, \ldots, p_n) \in \mathbf{R}^n \mid f(p_1, \ldots, p_n) = 0\}.$$

Such a set is called the *zero-locus of f*. The above notation is called set-builder notation, and should be read as "the set of all $(p_1, \ldots, p_n) \in \mathbf{R}^n$ such that $f(p_1, \ldots, p_n) = 0$". More generally, for any subset $S \subseteq \mathbf{R}[x_1, \ldots, x_n]$ (the symbol "$\subseteq$" should be read "is a subset of"), we can form the *zero-locus of S*:

$$Z(S) := \{(p_1, \ldots, p_n) \in \mathbf{R}^n \mid f(p_1, \ldots, p_n) = 0 \text{ for all } f \in S\}.$$

We can now formulate the following

Definition. A subset $X \subseteq \mathbf{R}^n$ is called an *algebraic variety over* $\mathbf{R}$ if there exists a subset $S \subseteq \mathbf{R}[x_1, \ldots, x_n]$ such that $Z(S) = X$.

A very nice fact is that if $X \subseteq \mathbf{R}^n$ is an algebraic variety, then there exist *finitely many* $f_1, \ldots, f_r \in \mathbf{R}[x_1, \ldots, x_n]$ such that $Z(f_1, \ldots, f_r) = X$. This is a consequence of the celebrated Hilbert's Basis Theorem.

Examples.

(1) Let $f(x,y) = x^2 + y^2 - 1 \in \mathbf{R}[x,y]$. Then $Z(f) \subseteq \mathbf{R}^2$ is the unit circle.
(2) Let $f(x,y) = y - x^2 \in \mathbf{R}[x,y]$. Then $Z(f) \subseteq \mathbf{R}^2$ is a parabola.
(3) Let $f(x,y) = xy - 1 \in \mathbf{R}[x,y]$. Then $Z(f) \subseteq \mathbf{R}^2$ is a hyperbola.
(4) Let $f(x,y,z) = x^2 + y^2 + z^2 - 1 \in \mathbf{R}[x,y,z]$. Then $Z(f) \subseteq \mathbf{R}^3$ is the unit sphere.
(5) Let $f(x,y) = y^2 - x^3 \in \mathbf{R}[x,y]$. Then $Z(f) \subseteq \mathbf{R}^2$ is the cuspidal cubic (see Figure 1 below).
(6) Let $f(x,y) = y^2 - x^2(x+1) \in \mathbf{R}[x,y]$. Then $Z(f) \subseteq \mathbf{R}^2$ is the nodal cubic (see Figure 2 below).
(7) Let $f(x,y,z) = y - x^2 \in \mathbf{R}[x,y,z]$, $g(x,y,z) = z - x^3 \in \mathbf{R}[x,y,z]$. Then $Z(f,g) \subseteq \mathbf{R}^3$ is the twisted cubic (see Figure 3 below).
(8) Let $f(x,y,z) = x^2 - y^2 z \in \mathbf{R}[x,y,z]$. Then $Z(f) \subseteq \mathbf{R}^3$ is the Whitney umbrella (see Figure 4 below).

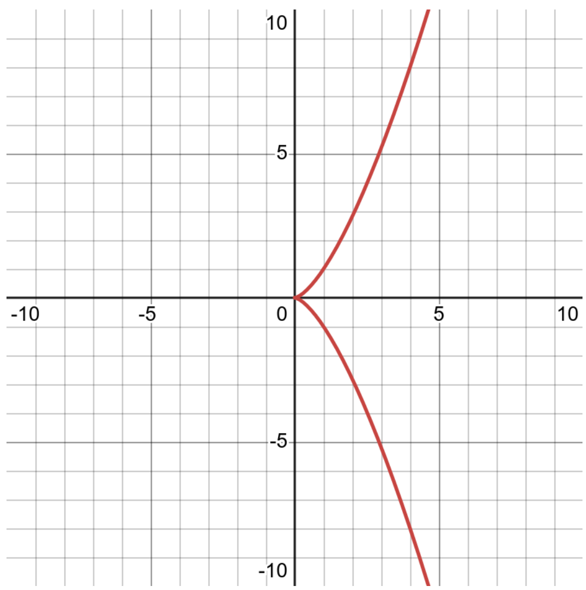

FIGURE 1. The cuspidal cubic.

MORPHISMS

We now move on to defining morphisms. We first define the notion of *regular function.*

Definition. Let $X \subseteq \mathbf{R}^n$ be an algebraic variety. A *regular function* $\varphi : X \to \mathbf{R}$ is one that is represented by a polynomial $f(x_1,\dots,x_n) \in \mathbf{R}[x_1,\dots,x_n]$. The set of all regular functions on X is denoted $A(X)$, and is called the *coordinate ring of* X.

What do I mean by "represented"? I mean that there is a polynomial $f(x_1,\dots,x_n) \in \mathbf{R}[x_1,\dots,x_n]$, viewed as a function, which agrees with φ on all points of X. The thing to keep in mind is that the same regular function can have different polynomial representatives.

We are now ready to define a morphism between two algebraic varieties.

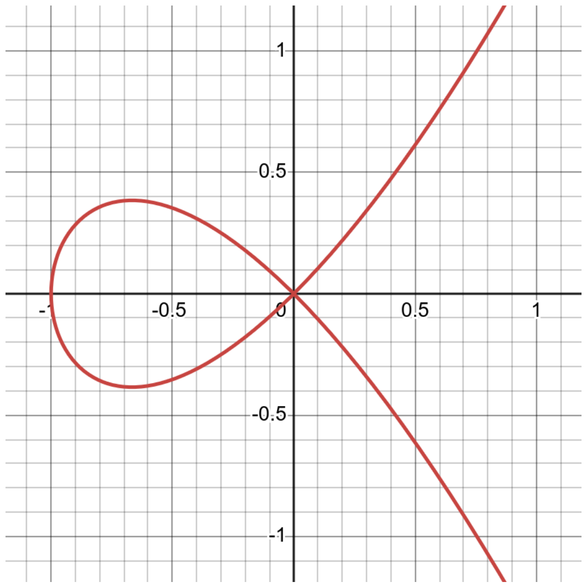

FIGURE 2. The nodal cubic.

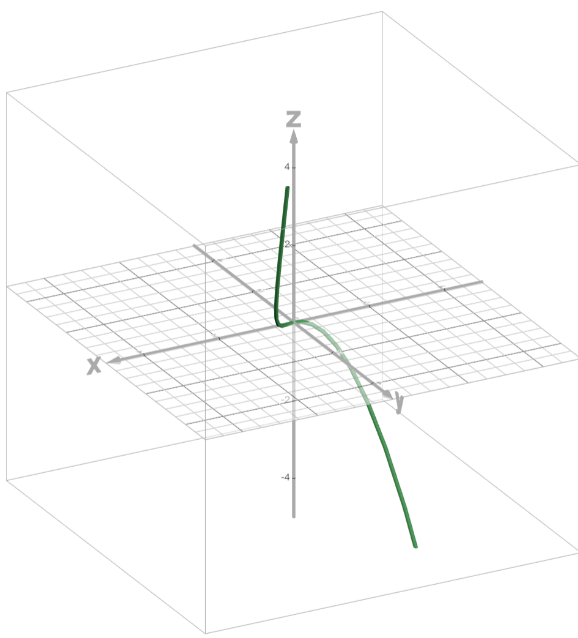

FIGURE 3. The twisted cubic.

Definition. Let $X \subseteq \mathbf{R}^n$, $Y \subseteq \mathbf{R}^m$ be algebraic varieties. A *morphism* $\varphi : X \to Y$ is given by $(a_1, \ldots, a_n) \mapsto (f_1(\underline{a}), \ldots, f_m(\underline{a}))$, where the f_i are regular functions on X and $\underline{a} = (a_1, \ldots, a_n)$.

In other words, morphisms can be thought of as "tuples of regular functions".

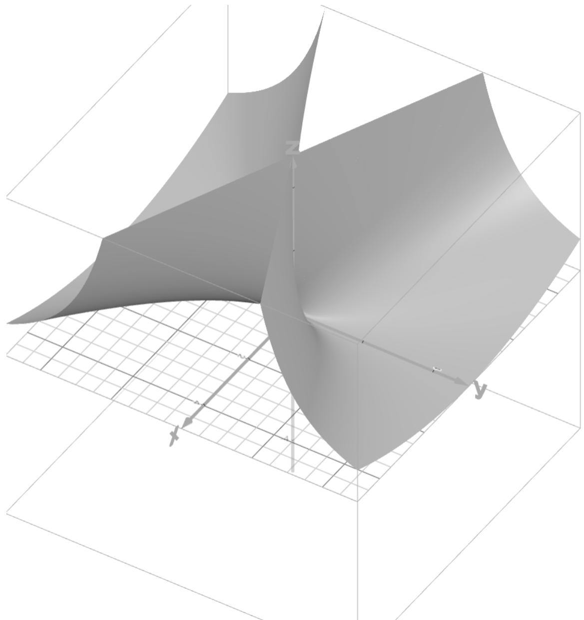

FIGURE 4. The Whitney umbrella.

HILBERT'S NULLSTELLENSATZ

Now that I have told you what algebraic varieties and morphisms between them are, we have answered the question "what is algebraic geometry about?". We can now move on to stating some theorems in algebraic geometry. The most fundamental theorem is *Hilbert's Nullstellensatz* (German for "Hilbert's Zero-Locus Theorem"), which was proved by David Hilbert in 1893. We will not state Hilbert's Nullstellensatz in full generality, but simply a particular case of it. In order to state the particular case, we must generalize the situation. Previously, we considered algebraic varieties over $\mathbf{R}$. Now, we consider algebraic varieties over the complex numbers $\mathbf{C}$. All of the previous definitions make sense if one replaces $\mathbf{R}$ by $\mathbf{C}$.

Let $X \subseteq \mathbf{C}^n$ be an algebraic variety. Denote by $I(X)$ the subset of $\mathbf{C}[x_1, \ldots, x_n]$ consisting of the polynomials that vanish at all points of X. We have the following theorem, which is a particular case of Hilbert's Nullstellensatz:

Theorem. *There is a one-to-one inclusion-reversing correspondence between*

$$\{\textit{algebraic varieties } X \subseteq \mathbf{C}^n\} \longleftrightarrow \{\textit{radical ideals } I \subseteq \mathbf{C}[x_1, \ldots, x_n]\}$$

given by $X \mapsto I(X)$.

What I mean by "inclusion-reversing" is that if $X \subseteq Y$ is an inclusion of varieties, then $I(Y) \subseteq I(X)$ is the corresponding inclusion of ideals. We will not define what an ideal is, nor what it means for an ideal to be radical. What is important is that ideals are algebraic objects, while varieties are geometric objects. Hilbert's Nullstellensatz is fundamental because it allows us to translate back and forth between algebra and geometry, giving us an algebro-geometric dictionary of sorts. It is, in a fundamental sense, the theorem that makes algebraic geometry possible.

Bézout's Theorem

We now move on to discussing Bézout's Theorem, which was proved by Étienne Bézout in 1779. We dicuss the planar case. To do this, we must first define projective space. For some intuition for projective space, imagine yourself standing on a straight railroad track, with your vision parallel to the rails of the track. The rails of the track are parallel, but from your perspective, they appear to intersect at the horizon. In projective space, the rails do actually intersect, and we say that the rails intersect at a "point at infinity".

Informally, projective 2-space over $\mathbf{C}$, denoted $\mathbf{CP}^2$, is defined to be the set $\mathbf{C}^2$ along with "points at infinity", such that any two distinct lines in $\mathbf{CP}^2$ intersect at exactly one point. Formally, $\mathbf{CP}^2$ is the set of points $\mathbf{C}^3 - \{(0,0,0)\}$, where we view scalar multiples of points as the same point, i.e., $(x,y,z) \sim (\lambda x, \lambda y, \lambda z)$ for all $\lambda \neq 0$. The notation we use for elements in $\mathbf{CP}^2$ is $[a : b : c]$, where $a, b, c \in \mathbf{C}$.

A *plane projective curve over* $\mathbf{C}$ is defined to be a set of the form

$$Z_+(f) := \{[p_0 : p_1 : p_2] \in \mathbf{CP}^2 \mid f(p_0, p_1, p_2) = 0\},$$

where $f \in \mathbf{C}[x,y,z]$ is a homogeneous polynomial, i.e., every monomial in f has the same degree. The *degree of* $Z_+(f)$ is the degree of the polynomial f. You can visualize projective plane curves as usual plane curves, keeping in mind that they include points at infinity.

We are now ready to state the following, which is a particular case of the planar case of Bézout's Theorem:

Theorem. *Let C, D be plane projective curves over $\mathbf{C}$ of degrees d, e respectively, with no component in common. Then the number of intersection points of C and D, counted with multiplicity, is exactly $d \cdot e$.*

The definition of the multiplicity of a point of intersection of two plane projective curves is rather technical. The intuitive idea is that, for example, if a curve C intersects itself at a point P, and another curve D intersects C at P, then we should consider the point P to have intersection multiplicity (with respect to C and D) greater than 1.

A Noble Lie

The "lie" referred to in the title of this section is not a statement of falsehood, but rather a lie by omission. Throughout the last two sections of this document, for the sake of simplicity, we were working with the complex numbers $\mathbf{C}$. In fact, everything we said throughout this document holds true for an arbitrary algebraically closed field. If you were attentive, you noticed that I stated particular versions of Hilbert's Nullstellensatz and the planar case of Bézout's Theorem. For the statements of the general versions, simply replace $\mathbf{C}$ with k, where k denotes an arbitrary algebraically closed field. In general, in classical algebraic geometry, we always work over an arbitrary algebraically closed field.

Also, there is a more general notion of variety than the one defined here. What I have called an algebraic variety is usually called an "affine algebraic variety", and the more general notion of a variety is that of a "quasi-projective variety".

Conclusion

Algebraic geometry is a field of study that is of central importance to modern mathematics. It has acquired the reputation of being extremely abstract. Because of this, it will naturally be difficult to explain algebraic geometry to a layman, but I hope I was able to provide you with even the slighest hint of what algebraic geometry is.