A SHORT EXPLANATION OF GÖDEL'S INCOMPLETENESS THEOREMS

Let F be a formal theory containing Robinson Arithmetic $\mathbf{Q}$. Then for any formula φ of F, we can associate with it $\ulcorner\varphi\urcorner$, the Gödel number of φ. Note that proofs also have Gödel numbers. Let $Prf_F(x, y)$ be the relation "x is the Gödel number of a proof in F of the formula with Gödel number y". It can be shown that Prf_F is strongly representable. The diagonalization lemma gives us that there exists a sentence G_F such that

$$F \vdash G_F \iff \forall x \, \neg Prf_F(x, \ulcorner G_F \urcorner). \tag{1}$$

Call a formal theory T ω-consistent if it is not the case that, for some formula $A(x)$, $T \vdash A(\underline{n})$ for all $n \in \mathbb{N}$ (where $\underline{n}$ is the numeral of n) and $T \vdash \neg\forall x \; A(x)$. We have the following

Theorem 1. *Let F be a formal theory containing $\boldsymbol{Q}$. Then there exists a sentence G_F such that*

a) *If F is consistent, then $F \nvdash G_F$.*
b) *If F is ω-consistent, then $F \nvdash \neg G_F$.*

Proof. Let G_F be the sentence in (1).

a) Suppose $F \vdash G_F$. Then there is a proof of G_F, say with Gödel number n. By the strong representability of Prf_F, we have that $F \vdash Prf_F(\underline{n}, \ulcorner G_F \urcorner)$. By $\exists$-introduction, $F \vdash \exists x \; Prf_F(x, \ulcorner G_F \urcorner)$. Hence, $F \vdash \neg\forall x \, \neg Prf_F(x, \ulcorner G_F \urcorner)$. However, by (1), $F \vdash \neg G_F$. Therefore, F is inconsistent.
b) Suppose F is ω-consistent. This implies that F is consistent. By the proof of a), we have that $F \nvdash G_F$. By the strong representability of Prf_F, $F \vdash \neg Prf_F(\underline{n}, \ulcorner G_F \urcorner)$ for all $n \in \mathbb{N}$. By ω-consistency, $F \nvdash \neg\forall x \, \neg Prf_F(x, \ulcorner G_F \urcorner)$. Therefore, $F \nvdash \neg G_F$, which was to be shown. □

Let the sentence $\underline{0} = \underline{1}$ be denoted by $\bot$. Define $Cons(F) := \neg\exists x \; Prf_F(x, \ulcorner \bot \urcorner)$. From the above, we also get the following

Theorem 2. *Let F be a formal theory containing Primitive Recursive Arithmetic $\boldsymbol{PRA}$. Then $F \nvdash Cons(F)$.*

Proof. The proof of Theorem 1 can be carried out within F to get $F \vdash Cons(F) \implies G_F$. If $F \vdash Cons(F)$, then $F \vdash G_F$, contradicting Theorem 1. □

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